PARTIAL DIFFERENTIAL EQUATION

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An equation which involves several independent variables denoted by $x, y, z, t, \ldots\ldots$ a dependent function u of these variables and its partial derivatives with respect to the independent variables such as

$$F(x, y, z, t, \ldots\ldots, u, u_x, u_y, u_z, u_t, \ldots\ldots u_{xx}, u_{yy}, \ldots\ldots u_{x_y} \ldots\ldots) = 0$$

is called a partial differential equation.

e.g. $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$ (Laplace equation)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{c^2}\frac{\partial^3 f}{\partial t^2}$$ (Wave equation)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{k}\frac{\partial f}{\partial t}$$

ORDER OF PDE

The order of the highest order partial derivative occurring in the equation is known as order of PDE.

e.g. Laplace equation, wave equations are of 2nd order partial differential equations

and $$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=0 \text{ is order one}$$

and $$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^3 f}{\partial y^3}=7\frac{\partial f}{\partial t}+\sin t \text{ is order three.}$$

PDEs OF FIRST ORDER IN TWO VARIABLES

(*i*) Let $z = f(x, y)$, then the equation which involved partial derivatives of **1st order** *i.e.* $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ is called **1st order partial** differential equation.

e.g. $f_1(x, y)\,\frac{\partial z}{\partial x} + f_2(x, y)\frac{\partial z}{\partial y} = f_3(x, y)$ is 1^{st} order PDE.

(ii) The PDE is called **linear** if the independent variable z and its partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ occur in linear degree (*i.e.* degree one) and none of two are multiplied with each other.

e.g. $$z \cdot \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = f(x, y)$$

and $$\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial y}\right)^2 + xy = 0$$

are non-linear whereas $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x + y$ is linear and also of 1st order.

(*iii*) The general first order partial differential equation is of the form $F(x, y, z, p, q) = 0$ where

$$p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}.$$

FORMATION OF FIRST ORDER PDEs

The relation $F(x, y, z, \alpha, \beta) = 0$ with two arbitrary constants α and β can be converted in PDE by differentiating F partially w.r.t. x and y then using all these equations and eliminating arbitrary constants α and β and this leads to the partial differential equation.

SOLUTION OR INTEGRAL OF A PARTIAL DIFFERENTIAL EQUATION

A solution or integral of a PDE is a relation between the variables which satisfies the given partial differential equation.

These are four types of solutions of a PDE :

(*i*) **Complete solution or complete integral** : If a relation $F(x, y, z, \alpha, \beta) = 0$ is obtained from the partial differential equation $f(x, y, z, p, q) = 0$; then the relation $F(x, y, z, \alpha, \beta) = 0$ is called complete integral or solution if it contains as many arbitrary constants as there are independent variables.

e.g. $z = (x + \alpha)(y + \beta)$ is a complete solution of PDE $z = pq$.

(*ii*) **Particular solution or particular integral :** If particular values are given to the arbitrary constants in the complete solution then the solution obtained is particular solution.

e.g. $z = (x + 5)(y + 7)$ is a particular solution of the partial differential equation $z = pq$.

(*iii*) **Singular solution or singular integral :** If $f(x, y, z, \alpha, \beta) = 0$ be the complete solution of the partial differential equation $f(x, y, z, p, q) = 0$ then the relation between x, y and z which is obtained by eliminating the arbitrary constants α and β between the equations

$$f(x, y, z, \alpha, \beta) = 0, \frac{\partial f}{\partial \alpha} = 0, \frac{\partial f}{\partial \beta} = 0$$

is known as singular solution of the PDE $f(x, y, z, p, q) = 0$ provided it satisfies the given equation.

(*iv*) **General solution or general integral :** If $f(x, y, z, \alpha, \beta) = 0$ be the complete solution of a partial differential equation $f(x, y, z, p, q) = 0$ and also if $\beta = \phi(\alpha)$ then $f(x, y, z, \alpha, \phi(x)) = 0$ is a one parameter family of the surfaces $f(x, y, z, p, q) = 0$. The relation between x, y and z obtained by eliminating arbitrary constants α between the equations $f(x, y, z, a, \phi(a)) = 0$ and $\frac{\partial f}{\partial a} = 0$ is known as a General solution of the PDE $F(x, y, z, p, q) = 0$, provided it satisfies the equation.

LAGRANGE LINEAR PARTIAL DIFFERENTIAL EQUATION

The linear partial differential equation of the form $P_p + Q_q = R$ where P, Q, R are functions of x, y, z is known as standard form of Lagrange's linear partial differential equation.

LAGRANGE'S METHOD TO SOLVE THE LINEAR PDE OF ORDER ONE

The general solution of linear partial differential equation $P_p + Q_q = R$; where P, Q and R are functions of x, y and z ; is $F(u, v) = 0$ where $F(u, v)$ is an arbitrary function of

$$u(x, y, z) = k_1$$

and

$$v(x, y, z) = k_2$$

which form a solution of $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$.

WORKING METHOD TO SOLVE THE LAGRANGE LINEAR EQUATION

(*i*) Write down the auxiliary equation for $P_p + Q_q = R$

i.e.
$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \qquad \ldots(1)$$

(*ii*) Find two independent integrals of the auxiliary equation (1) say $u = \alpha$ and $\nu = \beta$.

(*iii*) The general solution of $P_p + Q_q = R$ is $f(u, v) = 0$ or $u = \phi(v)$ where f or ϕ is an arbitrary function.

Let us take an example $p + q = \cos x$

Here $P = 1$, $Q = 1$, $R = \cos x$

$\therefore$ A.E. $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

becomes $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{\cos x}$

Taking 1st two members, we get

$$dx = dy$$

$$\Rightarrow \quad x = y + \text{constant}$$

$$\Rightarrow \quad (x - y) = k_1 \text{ (say)}$$

Taking 1st and 3rd members, we get

$$dx = \frac{dz}{\cos x}$$

$$\Rightarrow \quad dz = \cos x\, dx$$

$$\Rightarrow \quad z = \sin x + \text{constant}$$

$$\Rightarrow \quad (z - \sin x) = k_2 \text{ (say)}$$

Therefore the general integral is $f(x - y, z - \sin x) = 0$ where f is an arbitrary function.

CHARPIT'S METHOD FOR SOLVING FIRST ORDER PDEs

Let the given equation be

$$f(x, y, z, p, q) = 0 \quad \text{....(1)}$$

The Charpit's method consists of finding another first order partial differential equation

$$g(x, y, z, p, q, \alpha) = 0 \quad \text{...(2)}$$

where α is an arbitrary constant, such that

(*a*) equations (1) and (2) can be solved for p and q,

(*b*) $dz = p\,dx + q\,dy$ is integrable.

Therefore, the Charpit's auxiliary equations are

$$\frac{dx}{\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial f}{\partial q}} = \frac{dz}{p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}}$$

$$= \frac{dp}{-\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)} = \frac{dq}{-\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)} = \frac{dg}{0} \qquad \text{...(3)}$$

Solving Eqs. (2) and (3) for p and q and substituting in $dz = p\,dx + q\,dy$, then we get the complete solution $f(x, y, z, a, b) = 0$.

CAUCHY'S METHOD FOR SOLVING FIRST ORDER PDEs

Let the 1st order PDE is

$$F(x, y, z, p, q) = 0 \quad \dots(1)$$

then, the equations

$$\left.\begin{aligned} x'(t) &= \frac{\partial F}{\partial p}, y'(t) = \frac{\partial F}{\partial q}, z'(t) = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} \\ \text{and } p'(t) &= -\left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right), q'(t) = -\left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right) \end{aligned}\right\} \quad \dots(2)$$

are called Cauchy's characteristic equations of differential equation (1), and by using equation (2) we find the solution of Cauchy's problem.

PARTIAL DIFFERENTIAL EQUATION OF SECOND ORDER

A partial differential equation which contains at least one of the differential coefficients r, s, t with no higher order is known as second order partial differential equation, which is in the form of

$$F(x, y, z, p, q, r, s, t) = 0$$

where $p = \dfrac{\partial z}{\partial x}, q = \dfrac{\partial z}{\partial y}, r = \dfrac{\partial^2 z}{\partial x^2}, s = \dfrac{\partial^2 z}{\partial x \, \partial y}, t = \dfrac{\partial^2 z}{\partial y^2}$.

A partial differential equation of 2nd order is called linear if it is linear relative to the required function and all its derivatives that entre into the equation otherwise **non-linear**.

e.g. $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ is 2nd order and linear PDE.

$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial y^2}=z$ is linear of order two.

HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The differential equation of the type

$$a_0\frac{\partial^n z}{\partial x^n}+a_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2}+\ldots\ldots$$

$$+a_n\frac{\partial^n z}{\partial y^n}=f(x,y)$$

where a_0, a_1, a_2 ...... a_n are constants, is known as linear homogeneous partial differential equation.

Note : 1. A linear homogeneous partial differential equation is of the form where all the derivatives are of the same order.

e.g. $3\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 z}{\partial x\,\partial y} = e^{2x}$ is homogeneous linear partial differential equations with constant coefficients.

2. $\dfrac{\partial^3 z}{\partial x^3} + 5\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial z}{\partial y} = 3x + y$ is a partial differential equation of order 3 but it is not homogeneous.

3. For symbolic form write D for $\dfrac{\partial}{\partial x}$ and D' for $\dfrac{\partial}{\partial y}$.

$\therefore$ Differential equation written in Eq. (2) becomes

$$(D^3 + 5DD' + 2D') = 3x + y,$$

which is symbolic form.

4. The complete solution is

$$z = \text{C.F.} + \text{P.I.}$$

where Complementary Function (C.F.) is the solution of

$$F(D, D')z = 0$$

and Particular Integral (P.I.) is that value of z in terms of x, y which satisfies the given equation and contains no arbitrary constants.

5. If $z = f_1(x, y), z = f_2(x, y), \ldots\ldots z = f_n(x, y)$ are solutions of the homogeneous linear partial differential equation

$$F(D, D')z = 0$$

then

$$z = c_1 f_1(x, y) + c_2 f_2(x, y) + \ldots\ldots + c_n f_n(x, y)$$

is also a solution of

$$F(D, D')\, z = 0$$

where $c_1, c_2, \ldots\ldots c_n$ are arbitrary constants.

WORKING METHOD TO FIND THE SOLUTION OF HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS $f(D, D')z = 0$

Step I **:** Write down in symbolic form ;

$$D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}.$$

Step II **:** Write down A.E. by putting $D = m$, $D' = 1$ and solve.

Step III **:** Let $m_1, m_2, \ldots\ldots m_n$ are roots of A.E. then solution is

$$z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \ldots\ldots \phi_n(y + m_nx).$$

Remarks :

(*A*) If the A.E. has r roots equal (repeated), then C.F. is given by

$$z = \phi_1(y + m_1x) + x\phi_2(y + m_1x) + x^2\phi_3(y + m_1x) + \ldots\ldots x^{r-1}\,\phi_r(y + m_1x)$$

(*B*) If the A.E. has imaginary roots say $\alpha \pm i\beta$, then the corresponding terms *m* the general solution is

$$z = \phi_1[y + (\alpha + i\beta)x] + \phi_2[y + (\alpha - i\beta)x]$$

WORKING METHOD TO FIND THE SOLUTION OF HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION $f(D, D')z = f(x, y)$

For $f(D, D')z = f(x, y)$ the general solution is

$$z = \text{C.F.} + \text{P.I.}$$

(*A*) For C.F., see previous working rules.

(*B*) For P.I., working rule is as follow :

(*i*) P.I. $= \dfrac{f(x, y)}{F(D, D')}$. Factorise $F(D, D')$ and then resolving into partial functions or expanding in an infinite series

$$e.g. \text{P.I.} = \frac{x}{F(D, D')} = \frac{x}{(D^2 - a^2 D'^2)}$$

$$= \frac{1}{D^2}\left[1 - \frac{a^2 D'^2}{D^2}\right]^{-1} x = \frac{1}{D^2}(x)$$

$$= \frac{1}{D}\left(\frac{x^2}{2}\right) = \frac{x^3}{6}.$$

(*ii*) Where $f(x, y)$ is a function of $ax + by$ *i.e.* $f_1(ax + by)$

then $\text{P.I.} = \dfrac{f_1(ax+by)}{F(D, D')}$

when $F(D, D')$ is a rational integral homogeneous function of degree n.

Integrate $f_1(ax + by)$ n times w.r.t. $(ax + by)$ considered as one variable and then divide the result by $F(a, b)$.

In general $\dfrac{f_1(ax+by)}{(bD - aD')^r} = \dfrac{x^r}{b^r \lfloor r} f_1(ax+by)$

(*iii*) $\text{P.I.} = \dfrac{f(x, y)}{D - mD'} = \int f(x, a - mx)\, dx$ and replace a by $y + mx$ after integration.

NON-HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

The equation of the type

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^{n-1} z}{\partial x^{n-1}} + \ldots\ldots a_r \frac{\partial^{n-r} z}{\partial x\, \partial y^{(n-r-1)}} = f(x, y)$$

is non-homogeneous linear equations with constant coefficients.

(*a*) If $f(D, D')\,z = 0$ is as $(D - mD' - \alpha)\,z = 0$, then

$$z = e^{\alpha x} f(y + mx).$$

(*b*) If $f(D, D')\,z = 0$ is as $(D - mD' - \alpha)^r\,z = 0$ [*i.e.* repeated], then

$$z = e^{\alpha x} f_1(y + mx) + x e^{\alpha x} f_2(y + mx) + \ldots\ldots + x^{r-1} e^{\alpha x} f_r(y + mx)$$

(*c*) Let $F(D, D')\,z = 0$ be an irreducible non-homogeneous linear partial differential equation with constant coefficients then $z = ce^{ax + by}$ is a solution of $F(D, D')z = 0$ if $F(a, b) = 0$ and the constant c is an arbitrary.

(*d*) If in $F(D, D') = f(x, y)$; the function $f(x, y)$ is in the form of e^{ax+by}, then

$$\text{P.I.} = \frac{e^{ax+by}}{F(D, D')} = \frac{e^{ax+by}}{F(a, b)} \text{ provided } F(a, b) \neq 0.$$

[Put $D = a$ and $D' = b$]

(e) If in $F(D, D') = f(x, y)$; the function $f(x, y)$ is of the form $\sin(ax + by)$ or $\cos(ax + by)$, then

$$\text{P.I.} = \frac{f(x, y)}{F(D, D')}.$$

[Put $D^2 = -a^2$, $DD' = -ab$, $D'^2 = -b^2$]

(f) If in $F(D, D') = f(x, y)$; the function $f(x, y)$ is of the form $x^m y^n$ where m, n are +ve inegers then

$$\text{P.I.} = \frac{f(x, y)}{F(D, D')} = [F(D, D')]^{-1} (x^m y^n).$$

(g) If in $F(D, D') = f(x, y)$; the function $f(x, y)$ is in the form $(e^{ax + by} V)$, then

$$\text{P.I.} = \frac{(e^{ax+by} V)}{F(D, D')} = e^{ax+by} \frac{1}{F(D + a, D' + a)} (V).$$

Thank You