ORDINARY DIFFERENTIAL EQUATION

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A relation involving unknown functions and their derivatives w.r.t. one or more independent variable is called differential equation.

ORDINARY DIFFERENTIAL EQUATION

An equation which involves derivatives with respect to a single independent variable is called an ordinary differential equation

e.g. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0, \frac{dy}{dx} + xy = \tan x.$

ORDER OF A DIFFERENTIAL EQUATION

It is the order of the highest order derivative involved in a differential equation.

e.g. the order of $\frac{d^2y}{dx^2} + 7\left(\frac{dy}{dx}\right)^3 + 6y = 0$

is two as in 1st term y is differentiated twice which is the maximum derivative involved in the equation.

DEGREE OF A DIFFERENTIAL EQUATION

The exponent of the highest derivative involved in the differential equation is known as degree, provided all derivates involved in the differential equation are free from radicals and fractions.

e.g.

$$5\frac{d^2y}{dx} + 6\left(\frac{dy}{dx}\right)^5 + 7y = 0$$

has degree one because the highest order derivative is 2 which has exponent '1',

and degree of $\frac{dy}{dx} + \frac{5}{\frac{dy}{dx}} = 6$ is two because after simplification

it becomes $\left(\frac{dy}{dx}\right)^2 - 6\left(\frac{dy}{dx}\right) + 5 = 0.$

DIFFERENTIAL EQUATION (DE)

Differential Equation (DE)

- **Linear**
  - (*i*) Degree of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ should be one.
  - (*ii*) No two terms $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ will be in product.
- **Non-linear**
  - If any one of two conditions of linear differential equation is not true then it is non-linear differential equation.

For example :

$$\frac{dy}{dx} + 7y = 0,$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6 = 0,$$

$$\frac{d^3y}{dx^3} + 6y = 0,$$

$$\frac{d^2y}{dx^2} + 6x^2y = 0.$$

For example :

$$\frac{d^2y}{dx^2} + 7y^2 = 0.$$

[$\because$ degree of y is 2]

$$y\frac{dy}{dx} + 6x^2 = 0.$$

[$\because$ y and $\frac{dy}{dx}$ are in product]

SOLUTION OF A DIFFERENTIAL EQUATION

Simple meaning of solution of a differential equation means integration (integral of a differential).

Solution of DE

→ General Solution (GS) | Particular Solution (PS) | Singular Solution (SS)

General Solution (GS)

The solution of a differential equation which involves as many arbitrary constants as the order of the differential equation, is called the general solution (complete integral or complete primitive of a DE).

e.g. $x = a \cos(wt + b)$ is a general solution of the differential equation

$$\frac{d^2x}{dt^2} = -w^2x.$$

Particular Solution (PS)

The particular solution of a differential equation is that which is found from the general solution by giving particular values to the arbitrary constants [Particular Primitive]

$$x = 3\cos\left(wt + \frac{\pi}{3}\right)$$

is a particular solution of the differential equation $\frac{d^2x}{dt^2} = -w^2x.$

Singular Solution (SS)

A solution which cannot be obtained from the general solution by giving particular values to the arbitrary constants and which does not contain any arbitrary constant is called a singular solution. The singular solution of the DE

$$y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

is $x^2 + y^2 = a^2$ while its general solution is

$$y = cx + a\sqrt{a^2 + c^2}.$$

$\therefore$ The singular solution cannot be obtained from the general solution by assigning particular value to c.

FORMATION OF DIFFERENTIAL EQUATION

The differential equation is formed from the equation of a family of curves by differentiating it as many times as it has arbitrary constants and eliminating the arbitrary constants from the given equation and the equation obtained after differentiation.

For example **:** $y = a \cos(x + b)$, where a and b are two arbitrary constants

$\therefore$ Two times differentiation is required.

$\therefore \quad y_1 = -a \sin(x + b)$

$\Rightarrow \quad y_2 = -a \cos(x + b)$

$\therefore \quad y_2 = -y$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} + y = 0$ is the required differential equation

which has not no arbitrary constants and its order = 2 = number of arbitrary constants.

Geometrical Meanings

Geometrical meaning of the differential equation $\frac{dy}{dx} = f(x, y)$

$\frac{dy}{dx} = f(x, y)$ is a differential equation of 1st order and 1st degree.

$\therefore$ It represents a family of curves such that through every point, there passes one curve of the family.

Geometrical meaning of the complete solution of the differential equation $\frac{dy}{dx} = f(x, y)$

Let $g(x, y, c) = 0$ ...(1)

is the solution of

$$\frac{dy}{dx} = f(x, y) \quad ...(2)$$

where 'c' is an arbitrary constant.

Equation (1) represents one parameter family of curves such that through each point there passes one curve of the family.

DIFFERENTIAL EQUATIONS OF THE 1ST ORDER AND IST DEGREE

$$\frac{dy}{dx} = f(x, y)$$

or

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is called the differential equation of the 1st order and 1st degree. Differential equations of the 1st order and 1st degree are of the following types :

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(i) Variable separable (VS)

(ii) Reducible to variable separable (Red. to VS)

(iii) Homogeneous equations (HE)

(iv) Reducible to homogeneous equations (Red. to HE)

(v) Linear differential equations (LDE)

(vi) Reducible to linear differential equations (Red. to LDE)

BRIEF DESCRIPTION OF ABOVE MENTIONED TYPES

(*i*) **Variable separable :** $M(x, y)\, dx + N(x, y)\, dy = 0$ is changed into $f(x)\, dx + g(y)\, dy = 0$ and integrating.

(*ii*) **Reducible to variable separable :** If directly the linear differential equation is not possible to write in the form of VS, then substitution method helps us to convert into VS.

(*iii*) **Homogeneous equations :** The differential equation of the type $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ is called homogeneous equations if $f(x, y)$ and $g(x, y)$ are homogeneous expression.

e.g. $\frac{dy}{dx} = \frac{x + y}{x - y}$, $\frac{dy}{dx} = \frac{x^2 + y^2}{x + y}$, $\frac{dy}{dx} = \frac{x^3 + y^3}{x - y}$ etc.

Procedure to solve such equations :

(*i*) Put $y = vx$, where v is a function of x,

(*ii*) and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Therefore, given homogeneous equation reduced to variable separable form and solve it.

(*iv*) **Reducible to homogeneous equation :** The equation $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a_1x + b_1y + c_1}$ is not homogeneous but can be reducible to homogeneous.

(*a*) In $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a_1 x + b_1 y + c_1}$ if $\dfrac{a}{a_1} \neq \dfrac{b}{b_1}$ then

put $x = X + h, y = Y + k.$

$\therefore \; dx = dX, dy = dY$, we get

$$\frac{dY}{dX} = \frac{a(X + h) + b(Y + k) + c}{a_1(X + h) + b_1(Y + k) + c_1}$$

$$= \frac{aX + bY + (ah + bk + c)}{a_1 X + b_1 Y + (a_1 h + b_1 k + c_1)}$$

Select h, k s.t. $ah + bk + c = 0$ and $a_1 h + b_1 k + c_1 = 0.$

Solve for h, k and then use the technique of homogeneous form.

(*b*) In the differential equation

$$\frac{dy}{dx} = \frac{ax + by + c}{a_1x + b_1y + c_1}, \text{ if } \frac{a}{a_1} = \frac{b}{b_1}$$

then use substitution method and we get homogeneous form.

(*v*) **Linear differential equation (LDE) :** The form

$\frac{dy}{dx} + Py = Q$ is LDE

[where P and Q are functions of x.]

INTEGRATING FACTORS BY INSPECTION

Procedure to solve it :

$$\text{I.F.} = e^{\int P\,dx}$$

$\therefore$ Solution is $y\,.\,(\text{I.F.}) = \int (Q\,.\,\text{IF})\,dx + c.$

(*vi*) **Reducible to linear differential equation :**

$\dfrac{dy}{dx} + Py \neq Qy^n$ [Bernoulli's Equation] is not LDE.

$\therefore$ Dividing both sides by y^n and use substitution, then we get LDE.

EXACT DIFFERENTIAL EQUATIONS AND ITS INTEGRATING FACTORS

$M(x, y)\, dx + N(x, y)\, dy = 0$ is exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Solution of exact differential equation is

$$\int_{y \text{ as constant}} M\, dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

Integrating Factors

If any equation of the type $M\, dx + N\, dy = 0$ is not exact, then it can be made exact by multiplying it with a suitable function of x and y. This suitable function of x and y is called integrating factor (I.F.).

INTEGRATING FACTORS BY INSPECTION

Group of terms	*I.F.*	*Exact differential equation*
$x\,dy + y\,dx$	(*i*) $\frac{1}{x^2y^2}$	$\frac{x\,dy + y\,dx}{x^2y^2} = d\left(-\frac{1}{xy}\right)$
	(*ii*) $\frac{1}{x^2+y^2}$	$\frac{x\,dy + y\,dx}{x^2+y^2} = d\left[\frac{1}{2}\log(x^2+y^2)\right]$
$x\,dy - dy\,dx$	(*i*) $\frac{1}{x^2}$	$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$
	(*ii*) $\frac{1}{y^2}$	$\frac{x\,dy - y\,dx}{y^2} = -d\left(\frac{x}{y}\right)$
	(*iii*) $\frac{1}{xy}$	$\frac{x\,dy - y\,dx}{xy} = d\log\left\|\left(\frac{y}{x}\right)\right\|$
	(*iv*) $\frac{1}{x^2+y^2}$	$\frac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$

STANDARD RULES FOR FINDING INTEGRATING FACTORS

Rule 1 : If the equation $M\,dx + N\,dy = 0$ is homogeneous in x and y *i.e.* M and N are homogeneous function of the same degree in x and y, then $\dfrac{1}{Mx + Ny}$ is an integrating factor provided $Mx + Ny \neq 0$.

Rule 2 : If the equation $M\,dx + N\,dy = 0$ is of the form $yf(xy)\,dx + xg(xy)\,dx = 0$, then $\dfrac{1}{Mx - Ny}$ is an I.F. ($Mx - Ny \neq 0$).

Rule 3 : If $\left(\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}\right)$ is a function of x alone, say $f(x)$,

then $e^{\int (x)\, dx}$ is an I.F. of the equation
$M\, dx + N\, dy = 0$

Rule 4 : If $\left(\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}\right)$ is a function of y alone, say $f(y)$,

then $e^{\int (y)\, dx}$ is an I.F. of the equation
$M\, dx + N\, dy = 0.$

Rule 5 : If $\dfrac{a+u+1}{m} = \dfrac{b+\nu+1}{n}$ and $\dfrac{a'+u+1}{m'}$

$= \dfrac{b'+\nu+1}{n'}$ then $x^u y^\nu$ is an I.F. of the equation x^a
$y^b(my\,dx + nx\,dy) + x^{a'} y^{b'}(m'y\,dx + n'x\,dy) = 0.$

Thank You